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On finite group actions on an irreducible symplectic 4-fold

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1 Introduction

In this section, we will talk about background of our study. At first, we define an irreducible symplectic manifold.

Definition 1.1. *Let X be a compact Kähler manifold. When following two conditions are satisfied, we call X* **irreducible symplcetic manifold**.

1. X is simply connected.
2. $H^0(X,\Omega^2) = \mathbb{C}\langle\sigma_X\rangle$. where σ_X is an everywhere non-degenerate holomorphic 2-form.

In particular σ_X is said to be the symplectic form.

Remark 1.2. From existence of symplectic form, $\dim X$ is even, and a canonical bundle K_X is trivial. i.e.

$$\dim X = 2n,\ K_X \cong \mathcal{O}_X$$

We will introduce some famous examples. The easiest example is a K3 surface. Kodaira proved that a deformation equivalent class of K3 surface is unique. In higher dimensional case, there are only 4 types of deformation equivalent class which have been already known. Representative elements of each class are below.

Example

(i) n-pointed Hilbert scheme of K3 surface, $\mathrm{Hilb}^n(K3)$ ([Bea])

(ii) Generalized Kummer variety defined by Abelian surface A. We denote it by $\mathrm{Kum}^n(A)$ ([Bea]). Definition of $\mathrm{Kum}^n(A)$ is below.

$$\pi : \mathrm{Hilb}^{n+1}(A) \xrightarrow{\mu} \mathrm{Sym}^{n+1}(A) \xrightarrow{\Sigma} A$$

Where μ is Hilbert-Chow morphism. We define $\mathrm{Kum}^n(A) := \pi^{-1}(0)$.

(iii),(iv) O'Grady's six and ten dimensional example ([Ogr2],[Ogr])

We don't know whether above classes are all or not. By the way, Beaville and Donagi found another explicit example which is different from (i)~(iv). Let Y be a smooth cubic 4-fold, and let $F(Y)$ be all lines contained in Y. Then $F(Y)$ is an irreducible symplectic 4-fold ([B-D]). However, $F(Y)$ is deformation equivalent to a 2-pointed Hilbert scheme of a certain K3 surface $\mathrm{Hilb}^2(K3)$.

We investigated finite group actions on $F(Y)$ to make a new deformation equivalent class. We could not find it, but we met very interesting phenomena. We will introduce a part of them.

2 Preparation

In this section, we prepare some tools of our study.

Definition 2.1. *Let Y be a smooth cubic 4-fold. Let $F(Y)$ be all lines contained in Y. i.e.*

$$F(Y) := \{l \subset Y | l \cong \mathbb{P}^1, \deg l = 1\}$$

Remark 2.2. $F(Y)$ is a compact complex manifold whose dimension is 4.

Proposition 2.3 (Beauville-Donagi, [B-D]). *$F(Y)$is an irreducible symplectic manifold. In particular, $F(Y)$ is deformation equivalent to 2-pointed Hilbert scheme of a certain K3 surface* $\mathrm{Hilb}^2 K3$.

Let G be a finite group;

$$G \subset PGL(5),\ G \curvearrowright Y.$$

Since we want to make an irreducible symplectic manifold, first question is below.

Qustion 1. *When does $G \curvearrowright F(Y)$ preserve the symplectic form ?*

Let Γ be a universal family of $F(Y)$.

$$\Gamma := \{(l,y) \in F(Y) \times Y | l \ni y\}$$

There are two natural projections $p : \Gamma \to F(Y)$ and $q : \Gamma \to Y$. We define Abel-Jacobi map $\alpha : H^4(Y,\mathbb{C}) \to H^2(F(Y),\mathbb{C})$ as $\alpha(\omega) := p_* q^*(\omega)$. Abel-Jacobi map tells us whether G preserves the symplectic form or not.

$$\begin{array}{ccc} H^4(Y,\mathbb{C}) & \xrightarrow[\alpha]{\simeq} & H^2(F(Y)) \\ \uparrow & & \uparrow \\ H^{3,1}(Y) & \longrightarrow & H^{2,0}(F(Y)) \\ \| & & \| \\ \mathbb{C}\langle Res\frac{\Omega}{f^2}\rangle & \longrightarrow & \mathbb{C}\langle\sigma_{F(Y)}\rangle \end{array}$$

Where Ω is five form on $\mathbb{C}^6$ defined as $\Omega := \sum_{i=0}^5 (-1)^i z_i dz_0 \wedge \cdots \hat{dz_i} \cdots \wedge dz_5$. Since Abel-Jacobi map α is G-equivariant, we get a following lemma.

Answer of Question 1.

Lemma 2.4. *Notations as above.*

$$G \text{ preserves } \sigma_{F(Y)} \iff G \text{ preserves } Res\frac{\Omega}{f^2}$$

In general, $F(Y)/G$ may have singular points. So, we have to take resolution of $F(Y)/G$. We require that a resolution of $F(Y)/G$ has a symplectic form. So, second question is

Qustion 2. *When does $F(Y)/G$ have a crepant resolution $\widetilde{F(Y)/G}$?*

It is easy to find group actions $G \curvearrowright F(Y)$ which preserve the symplectic form, but it's difficult to find group actions such that $\widetilde{F(Y)/G}$ exists.

We have two examples of "good" actions. In this poster, our topic is one of them.

3 First example

First example was found by Namikawa.

Assumption

We consider special cubic 4-fold Y;

$$Y := \{f(z_0,z_1,z_2) + g(z_3,z_4,z_5) = 0\},$$

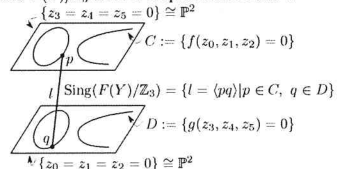
where f and g are homogeneous polynomial with degree 3.
Assume that $G = \mathbb{Z}_3$(order three cyclic group) and τ is a generator of G: $G = \langle\tau\rangle \simeq \mathbb{Z}_3$. We consider following group action;

$$\tau \curvearrowright \mathbb{P}^5 \text{ as } (z_0 : z_1 : z_2 : \zeta z_3 : \zeta z_4 : \zeta z_5),$$

where $(z_0 : \cdots : z_5)$ is homogeneous coordinate of $\mathbb{P}^5$, and $\zeta = \exp(\frac{2\pi\sqrt{-1}}{3})$. In particular, G acts on Y.

From Lemma 2.4, we know that the induced action on $F(Y)$ preserves the symplectic form. Next we consider singular points of $F(Y)/\mathbb{Z}_3$.

Does $F(Y)/\mathbb{Z}_3$ have a crepant resolution ?



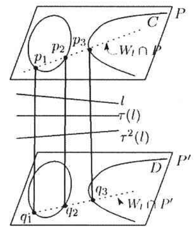
C and D are elliptic curves defined as above. $C \cup D$ is fixed locus of $\mathbb{Z}_3 \curvearrowright Y$. Singular locus of $F(Y)/\mathbb{Z}_3$ is isomorphic to $C \times D$. Since $\mathbb{Z}_3$ preserves the symplectic form, $F(Y)/\mathbb{Z}_3$ has A_2 singularities along $C \times D$. So, $\widetilde{F(Y)/\mathbb{Z}_3}$ does exist. What is $\widetilde{F(Y)/\mathbb{Z}_3}$?

Answer

Proposition 3.1 ([Nam]). *Notations as above. $\widetilde{F(Y)/\mathbb{Z}_3}$ is birational to* $\mathrm{Kum}^2(C \times D)$

Remark 3.2. If two irreducible symplectic manifold X and X' are birational, then X and X' are deformation equivalent. So, $\widetilde{F(Y)/\mathbb{Z}_3}$ is not new example.

Proof. We construct birational map $\psi : \widetilde{F(Y)/\mathbb{Z}_3} \dashrightarrow \mathrm{Kum}^2(C \times D)$. Instant picture of ψ is below.



$$\psi : \{l, \tau(l), \tau^2(l)\} \mapsto \{(p_i, q_i)\}_{i=1}^3$$

Let $\{l, \tau(l), \tau^2(l)\}$ be in $\widetilde{F(Y)/\mathbb{Z}_3}$. Let W_l be a liner space spanned by l, $\tau(l)$ and $\tau^2(l)$.

$$W_l := \langle l, \tau(l), \tau^2(l)\rangle \simeq \mathbb{P}^3.$$

Suppose that $P = \{z_3 = z_4 = z_5 = 0\}, P' = \{z_0 = z_1 = z_2 = 0\}$. If we choose l in general, we may assume that $S := W_l \cap Y$ is a smooth cubic surface. There are 27 lines in S(classical results). From the configuration of 27 lines, we know that there exist three lines m_1, m_2, m_3 such that each m_i meets $l, \tau(l), \tau^2(l)$ like above picture. Each $m_i (i = 1,2,3)$ meets C (resp D) at one point. So we set notations as $p_i = m_i \cap C, q_i := m_i \cap D$. Since three points $\{p_1,p_2,p_3\}$(resp. $\{q_1,q_2,q_3\}$) are coliner, $p_1 + p_2 + p_3 = 0 \in C$(resp. $q_1 + q_2 + q_3 = 0 \in D$). So we have a pair of three points $\{(p_i,q_i)\}_{i=1}^3$. □

Where is the indeterminacy of ψ ?

We determine the indeterminacy of ψ and ψ^{-1}. Indeterminacy of ψ is

$$\{[l] := \{l, \tau(l), \tau^2(l)\} \in \widetilde{F(Y)/\mathbb{Z}_3}\ ||l| \text{ spans } \mathbb{P}^2\}.$$

This locus is 18 copies of $\mathbb{P}^2$. Indeterminacy of ψ^{-1} are two types. First one is

$$P_{(I)} := \{\{(p,q_1),(p,q_2),(p,q_3)\} \in \mathrm{Kum}(C \times D) | 3p = 0\}$$

Second one is

$$P_{(II)} := \{\{(p_1,q),(p_2,q),(p_3,q)\} \in \mathrm{Kum}(C \times D) | 3q = 0\}$$

$P_{(I)}$ and $P_{(II)}$ are isomorphic to 9 copies of $\mathbb{P}^2$.

Let X and X' be an irreducible symplectic **4-fold**. It is known that any birational map from X to X' is decomposed into Mukai-flop. We have a following theorem.

Theorem 3.3. *The indeterminacy of ψ can be resolved by Mukai-flop on 18 copies of $\mathbb{P}^2$.*

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